Back to the roots: rooting multivariate polynomials with numerical linear algebra and nD system theory

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Everybody has been taught how to solve elementary polynomials in one unknown x, linear ones of the form b.x+c=0, or quadratic ones like a.x^2+b.x+c=0, when a,b and c are given. From degree 5 or higher, there do not exist closed formulas to find all the zeros as a function of the coefficients and one has to resort to numerical algorithms.

Even more complicated is to find several unknowns (x,y,z,....) that are roots of a set of multivariate polynomials with real coefficients. This problem is one of the corner stones of 'algebraic geometry', the achievements of which have been truly mind boggling, leading to Andrew Wiles' proof of Fermat's Last Theorem in number theory, or to Shing-Tung Yau's proof of the Calabi-Yau conjecture in string theory.

In this talk, we return to the more prosaic 'original' roots of the field, which was born in the analysis of roots of univariate polynomials (Gauss' Fundamental Theorem of Linear Algebra, Galois' 'Impossibility' Theory). We will develop new theory and numerical algorithms to 'root' sets of multivariate polynomials. This key problem arises in an abundant number of applications in the sciences, mathematics and engineering (advanced geometry, chemical equilibrium systems, Nash equilibria in game theory, cryptography, kinematics and robotics, Markov chain modeling, dynamical system modeling, optimal control theory, etc.).

The real nature of the problem is quite simple: Finding all the common roots of a system of multivariate polynomials is essentially equivalent to an eigenvalue problem in one variable, a result already known to Sylvester. In this talk, we analyse the properties of the so-called Macaulay matrix, and in doing so we reconcile the world of algebraic geometry with insights and concepts from multidimensional state space system theory, in which state space models are used to describe the dynamic behavior of systems in n independent variables (e.g. 3 spatial and 1 temporal variable), such as the temperature distribution in a room as a function of time. We will explain how n-D realization theory leads to eigenvalue problems that allow to determine the roots of the system. These insights result in numerical linear algebra algorithms, that efficiently calculate one, some or all of the roots.